Bounded hypergeometric functions and 2-parameter deformations of multiplicities

Angela Pasquale

Institut Élie Cartan de Lorraine
Université de Lorraine – Metz

Joint work with
E. K. Narayanan (Indian Institute of Sciences, Bangalore)

Journées SL2R
Théorie des Représentations et Analyse Harmonique

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Helgason-Johnson's theorem: the bounded spherical functions

G/K = Riemannian symmetric space of the noncompact type

G = connected noncompact semisimple Lie group with finite center, e.g. $SL_2(\mathbb{R})$
K = maximal compact subgroup of G, e.g. $SO_2(\mathbb{R})$

Spherical functions = (normalized) K-invariant joint eigenfunctions of the commutative algebra $\mathbb{D}(G/K)$ of G-invariant differential operators on G/K

$\rightsquigarrow$ matrix coefficients (for the K-fixed vector) of spherical principal series reprs
$\rightsquigarrow$ building blocks of the K-invariant harmonic analysis on G/K

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition of the Lia algebra of G
$\mathfrak{a} \subset \mathfrak{p}$ maximal abelian subspace (Cartan subspace)
Σ = (restricted) roots of $(\mathfrak{g}, \mathfrak{a})$
W = Weyl group of Σ

$\rightsquigarrow$ spherical functions are parametrized by $\mathfrak{a}_{\mathbb{C}}^*$ (modulo W)

Σ^+ = choice of positive roots in Σ
m_α = multiplicity of the root $\alpha \in \Sigma$
$\rho = 1/2 \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$

Harish-Chandra's integral formula: $\varphi_\lambda(g) = \int_K e^{(\lambda-\rho)(H(gk))}\, dk\,, \quad g \in G\,,$
where $H(x) \in \mathfrak{a}$ is the Iwasawa component of $x \in G = KAN$
Then: $\varphi_{w\lambda} = \varphi_\lambda$ for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $w \in W$.

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Question:

For which parameters $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ the spherical function

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is bounded?

$C(\rho)$ = convex hull in $\mathfrak{a}^*$ of $\{w\rho : w \in W\}$

Theorem (Helgason & Johnson, 1969)

The spherical function φ_λ (with $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$) is bounded if and only if $\lambda \in C(\rho) + i\mathfrak{a}^*$. In this case, $|\varphi_\lambda(g)| \leq 1$ for all $g \in G$.

Applications:

$L^1(G//K) = \{f : G \to \mathbb{C} : L^1 \text{ and } K\text{-biinvariant: } f(k_1 g k_2) = f(g),\ \forall g \in G,\ \forall k_1, k_2 \in K\}$

- $L^1(G//K)$ is a commutative Banach algebra with respect to convolution. The continuous characters of $L^1(G//K)$ are the maps
  $$f \in L^1(G//K) \longmapsto \int_G f(g)\varphi_\lambda(g)\, dg$$
  where φ_λ is a bounded spherical function, i.e. the bdd spherical functions parametrize the maximal ideal space of $L^1(G//K)$
- Spherical Fourier transform of $f \in L^1(G//K)$:
  $$(\mathcal{F}f)(\lambda) = \int_G f(g)\varphi_\lambda(g)\, dg\,, \qquad \lambda \in \mathfrak{a}_{\mathbb{C}}^*.$$
  φ_λ is an entire function of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. If $\exists U$ open in $\mathfrak{a}_{\mathbb{C}}^*$ such that φ_λ is bounded for $\lambda \in U$, then $\mathcal{F}f(\lambda)$ is holomorphic in U.

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Heckman-Opdam's hypergeometric functions

- The symmetric space G/K is replaced by a triple $(\mathfrak{a}, \Sigma, m)$ where:
  $\mathfrak{a}$ = finite dim. Euclidean $\mathbb{R}$-vector space, inner product $\langle \cdot, \cdot \rangle$
  Σ = root system in $\mathfrak{a}^*$, with Weyl group W
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$(\mathfrak{a}, \Sigma, m)$ is *geometric* if from a Riemannian symm space of the noncompact type

- Commutative family $\mathbb{D}$ of differential operators associated with $(\mathfrak{a}, \Sigma, m)$:

For $x \in \mathfrak{a}$ the *Cherednik operator* T_x is the difference-reflection operator on $\mathfrak{a}$ (or $\mathfrak{a}_\mathbb{C}$) defined for $f \in C^\infty(\mathfrak{a})$ and $H \in \mathfrak{a}$ by

$$T_x f(H) = \partial_x f(H) + \sum_{\alpha \in \Sigma^+} m_\alpha \alpha(x) \frac{f(H) - f(r_\alpha H)}{1 - e^{-2\alpha(H)}} - \rho(m)(x) f(H)$$

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$\{x \in \mathfrak{a} \mapsto T_x\}$ commutative $\Rightarrow$ extends as algebra homom $\{p \in S(\mathfrak{a}_\mathbb{C}) \mapsto T_p\}$

- If $p \in S(\mathfrak{a}_\mathbb{C})^W$, then $D_p := T_p|_{C^\infty(\mathfrak{a})^W}$ is a diff operator on $\mathfrak{a}$ (or $\mathfrak{a}_\mathbb{C}$).

$\mathbb{D}(\mathfrak{a}, \Sigma, m) := \{D_p : p \in S(\mathfrak{a}_\mathbb{C})^W\}$

Example: $\mathbb{D}(\mathfrak{a}, \Sigma, m) = \mathbb{D}(G/K)|_\mathfrak{a}$ if $(\mathfrak{a}, \Sigma, m)$ is geometric.

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$$T_x f(H) = \partial_x f(H) + \sum_{\alpha \in \Sigma^+} m_\alpha \alpha(x) \frac{f(H) - f(r_\alpha H)}{1 - e^{-2\alpha(H)}} - \rho(m)(x) f(H)$$

where: $r_\alpha =$ reflection across $\ker \alpha$,
$\rho(m) = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$.

$\{x \in \mathfrak{a} \mapsto T_x\}$ commutative $\Rightarrow$ extends as algebra homom $\{p \in S(\mathfrak{a}_\mathbb{C}) \mapsto T_p\}$

- If $p \in S(\mathfrak{a}_\mathbb{C})^W$, then $D_p := T_p|_{C^\infty(\mathfrak{a})^W}$ is a diff operator on $\mathfrak{a}$ (or $\mathfrak{a}_\mathbb{C}$).
  $\mathbb{D}(\mathfrak{a}, \Sigma, m) := \{D_p : p \in S(\mathfrak{a}_\mathbb{C})^W\}$

Example: $\mathbb{D}(\mathfrak{a}, \Sigma, m) = \mathbb{D}(G/K)|_\mathfrak{a}$ if $(\mathfrak{a}, \Sigma, m)$ is geometric.

- *Hypergeometric function* of spectral parameter $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$:

unique W-invariant analytic function F_λ on $\mathfrak{a}$ which satisfies the system of diff eqs

$$D_p F_\lambda = p(\lambda) F_\lambda \,, \qquad p \in S(\mathfrak{a}_{\mathbb{C}})^W \,,$$
$$F_\lambda(0) = 1$$

Then: $F_{w\lambda} = F_\lambda$ for all $w \in W$.

Examples:

(1) $(\mathfrak{a}, \Sigma, m)$ geometric: $\mathfrak{a} \equiv \exp \mathfrak{a} \cdot o \subset G/K$

$F_\lambda \equiv \varphi_\lambda$ Harish-Chandra's spherical function of spectral parameter $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$

(2) rank-one (i.e. $\dim_{\mathbb{R}} \mathfrak{a} = 1$): Jacobi function of 2nd kind

$F_\lambda(x) = {}_2F_1\left(\frac{m_\alpha/2+m_{2\alpha}+\lambda}{2}, \frac{m_\alpha/2+m_{2\alpha}-\lambda}{2}; \frac{m_\alpha/2+m_{2\alpha}+1}{2}; -\sinh^2 x\right)$

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Theorem (Anker (1987, $\lambda = 0$), Schapira (2008), Narayanan-P.-Pusti (2014))

Let $\lambda \in \overline{(\mathfrak{a}^*)^+}$. Then for all $x \in \overline{\mathfrak{a}^+}$

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Heckman-Opdam's hypergeometric functions

(Koornwinder, Heckman & Opdam, Cherednik,...)

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Spherical functions on a Riem. symmetric space G/K of the noncompact type

(Harish-Chandra, Helgason,...)

$\cap$ $[\tau = 1]$

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τ-spherical functions on the homog. vector bundle over G/K associated with τ

(τ= irr. unitary rep of K)

(Godement, Harish-Chandra,...)

Two questions:

- Restricted to A, can τ-spherical functions for arbitrary τ be expressed in terms of Heckman-Opdam hyergeometric functions?
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τ-spherical functions on G

(τ, V_τ)= finite dim irreducible representation of K

$f : G \to \mathrm{End}(V_\tau)$ is *τ-radial* if $f(k_1 g k_2) = \tau(k_2^{-1}) f(g) \tau(k_1^{-1})$, $\forall g \in G$, $\forall k_1, k_2 \in K$

$L^1(G//K;\tau) = \{f : G \to \mathrm{End}(V_\tau) : L^1$ and τ-radial$\}$, convolution algebra

Ex.: $L^1(G//K)$ if $\tau = 1$.

(G, K, τ) is a *Gelfand triple* if $L^1(G//K;\tau)$ is commutative.

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τ-spherical functions = (normalized) τ-radial joint eigenfunctions of $\mathbb{D}(G/K;\tau)$.

Other approaches are possible when (G, K, τ) not Gelfand: via representation theory (Harish-Chandra, Warner, Varadarajan...) or via $\mathbb{D}(G/K;\tau)$ (Olbrich).

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(G, K, τ) is a *Gelfand triple* if $L^1(G//K;\tau)$ is commutative.

$C^\infty(G/K;\tau) = \{F : G \to V_\tau : C^\infty$ and $F(gk) = \tau(k^{-1})F(g)$, $^\forall g \in G$, $^\forall k \in K\}$
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In this case:

τ-spherical functions = (normalized) τ-radial joint eigenfunctions of $\mathbb{D}(G/K;\tau)$.

Other approaches are possible when (G, K, τ) not Gelfand: via representation theory (Harish-Chandra, Warner, Varadarajan...) or via $\mathbb{D}(G/K;\tau)$ (Olbrich).

$G = KAK \rightsquigarrow \tau$-radial functions are uniquely determined by restriction to A.

τ-spherical functions on G

(τ, V_τ)= finite dim irreducible representation of K

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- Case-by-case (using the classification of small K-types, which they completed).
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- "Written in terms of" means that Heckman-Opdam's hypergeometric functions are multiplied by suitable products of cosh- and sinh-like factors, depending on the two root systems.

Remarkable, but hard to deal with in a unified way.

- When Σ^τ is a BC_n root system, it is possible to unify all these specific cases as elements of a 2-parameter deformation of Heckman-Opdam's hypergeometric functions.

τ-spherical functions for a small K-type τ

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2-parameter deformations of positive multiplicites

Σ = root system in $\mathfrak{a}$ of type BC_r

W = Weyl group of Σ

Three W-group orbits in Σ, distinguished by root lenght (short, middle and long roots)

$\Sigma^+ = \Sigma_{\rm s}^+ \sqcup \Sigma_{\rm m}^+ \sqcup \Sigma_{\rm l}^+$ where

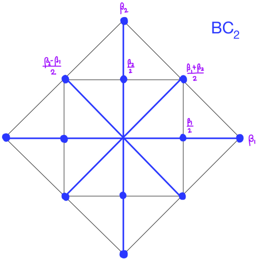
$\Sigma_{\rm s}^+ = \{\frac{\beta_j}{2} : 1 \leq j \leq r\}, \quad \Sigma_{\rm m}^+ = \{\frac{\beta_j \pm \beta_i}{2} : 1 \leq i < j \leq r\}, \quad \Sigma_{\rm l}^+ = \{\beta_j : 1 \leq j \leq r\}.$

Three values of multiplicities $m = (m_{\rm s}, m_{\rm m}, m_{\rm l})$. Suppose these three values are ≥ 0.

Consider C_r as a BC_r root system with $m_{\rm s} = 0$ and A_1 as a BC_1 with $m_{\rm s} = m_{\rm m} = 0$.

For any two real parameters $\ell, \widetilde{\ell}$ we define a deformation $m(\ell, \widetilde{\ell})$ of m as follows:

$$m_\alpha(\ell, \widetilde{\ell}) = \begin{cases} m_{\rm s} + 2\ell & \text{if } \alpha \in \Sigma_{\rm s} \\ m_{\rm m} + 2\widetilde{\ell} & \text{if } \alpha \in \Sigma_{\rm m} \\ m_{\rm l} - 2\ell & \text{if } \alpha \in \Sigma_{\rm l}\,. \end{cases}$$



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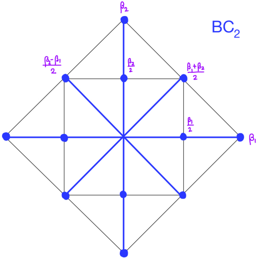

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$(\ell, \widetilde{\ell})$-hypergeometric functions

$u(x) = \prod_{j=1}^{r} \cosh\left(\frac{\beta_j(x)}{2}\right)$

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- Commutative family of differential operators associated with $(\mathfrak{a}, \Sigma, m, (\ell, \widetilde{\ell}))$:

For $x \in \mathfrak{a}$ the $(\ell, \widetilde{\ell})$-*Cherednik operator* T_x is the difference-reflection operator on $\mathfrak{a}$ (or $\mathfrak{a}_{\mathbb{C}}$) defined by

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Examples

(1) Write $m(l)$ and $F_{\ell,\lambda}$ if $\widetilde{\ell} = 0$.
If $m = (m_{\rm s}, m_{\rm m}, m_{\rm l} = 1)$ is a geometric multiplicity corresponding to a Hermitian symmetric space G/K, then the

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$\rightsquigarrow$ Σ and m are not necessarily those naturally associated with G/K.

Example: $G = \mathrm{Spin}(2r, 1)$ (double cover of $\mathrm{SO}(2r, 1)$), $K = \mathrm{Spin}(2r)$
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are the restrictions to $A \equiv \mathfrak{a}$ of the spherical τ_ℓ-functions, where $\dim \tau_\ell = 1$ (Shimeno, 1994).

(2) More generally, every τ-spherical function for G/K where τ is a small K-type and G/K has restricted root system of type BC is a $(\ell, \widetilde{\ell})$-hypergeometric function for suitable choices of $(\Sigma, m, (\ell, \widetilde{\ell}))$ where $m = (m_{\rm s}, m_{\rm m}, m_{\rm l})$ is a positive mutliplicity on Σ.

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$\rightsquigarrow$ Σ and m are not necessarily those naturally associated with G/K.

Example: $G = \mathrm{Spin}(2r, 1)$ (double cover of $\mathrm{SO}(2r, 1)$), $K = \mathrm{Spin}(2r)$
$\tau_1^{\pm}$ the positive and negative spin representations, i.e. the irreducible representation of K of heighest weight $(1/2, \dots, 1/2, \pm 1/2)$.
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Needed:

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- Estimates for the $F_\lambda(m(\ell,\widetilde{\ell}))$ and for $F_{\ell,\widetilde{\ell},\lambda}(m)$

  $m(\ell,\widetilde{\ell}) = (m_{\mathrm{s}} + 2\ell, m_{\mathrm{m}} + 2\widetilde{\ell}, m_{\mathrm{l}} - 2\ell)$.
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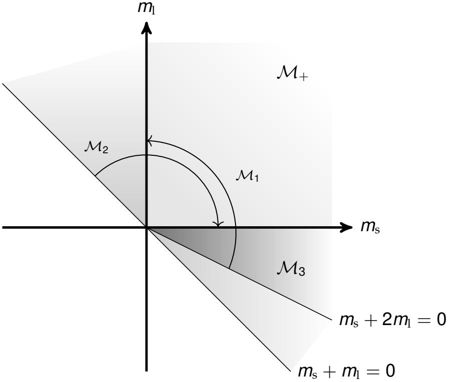
Multiplicity functions

$\mathcal{M}$ = set of $\mathbb{R}$-valued multiplicity functions $m = (m_{\mathrm{s}}, m_{\mathrm{m}}, m_{\mathrm{l}})$ on Σ of type BC_r.

$\mathcal{M}_+ = \{m \in \mathcal{M} : m_\alpha \geq 0 \text{ for every } \alpha \in \Sigma\}$ (non-negative multiplicities)
$\mathcal{M}_0 = \{m \in \mathcal{M} : m_{\mathrm{m}} \geq 0, m_{\mathrm{s}} + m_{\mathrm{l}} \geq 0\}$
$\mathcal{M}_1 = \{m \in \mathcal{M} : m_{\mathrm{m}} > 0, m_{\mathrm{s}} > 0, m_{\mathrm{s}} + 2m_{\mathrm{l}} > 0\}$ (standard multiplicities)
$\mathcal{M}_2 = \{m \in \mathcal{M} : m_{\mathrm{m}} \geq 0, m_{\mathrm{l}} \geq 0, m_{\mathrm{s}} + m_{\mathrm{l}} \geq 0\}$
$\mathcal{M}_3 = \{m \in \mathcal{M} : m_{\mathrm{m}} \geq 0, m_{\mathrm{l}} \leq 0, m_{\mathrm{s}} + 2m_{\mathrm{l}} \geq 0\}$.



- $\mathcal{M}_1 = (\mathcal{M}^+ \cup \mathcal{M}_3)^0$
- $\mathcal{M}_0$ is the natural subset of $\mathcal{M}$ on which both $G_\lambda(m)$ and $F_\lambda(m)$ are def for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.
- The asymptotics of $F_\lambda(m)$ determined by Narayanan-Pusti-P. (2014) for $m \in \mathcal{M}_+$ hold also for $m \in \mathcal{M}_1$.
- The real version of Opdam's estimates established (in much stronger form) by Ho-Ólafsson (2014) for $m \in \mathcal{M}_2$ hold also for $m \in \mathcal{M}_3$.
- On $\mathcal{M}_3$ on a $m_{\mathrm{l}} \leq 0$ but $\rho(m) \in \overline{(\mathfrak{a}^*)^+}$.

Estimates of $F_\lambda(m, x)$ for $x \in \mathfrak{a}$

Theorem

Let $m \in \overline{\mathcal{M}}_1 = \mathcal{M}_+ \cup \mathcal{M}_3$. Then the following properties hold on $\mathfrak{a}$:

(a) *For all $\lambda \in \mathfrak{a}^*$ the functions $F_\lambda(m)$ are real and strictly positive.*

(b) $|F_\lambda(m)| \leq F_{\mathrm{Re}\,\lambda}(m)$.

(c) *For all $\lambda \in \mathfrak{a}^*$, $\mu \in \overline{(\mathfrak{a}^*)^+}$ and $x \in \mathfrak{a}$*

$$F_{\lambda+\mu}(m; x) \leq F_\mu(m; x) e^{\max_{w \in W} (w\lambda)(x)} .$$

Schapira, 2008: (a) and (b) for $\mathcal{M}_+$
Extended to $\mathcal{M}_3$ by modifying his arguments.

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Extended to $\mathcal{M}_3$ by modifying their arguments.

- Sharp asymptotics and Harish-Chandra estimates hold on $\mathcal{M}_1$.
- Can apply these estimates/asymptotics to $F_{\ell,\widetilde{\ell},\lambda}(m) = u^{-\ell} v^{-\widetilde{\ell}} F_\lambda(m(\ell, \widetilde{\ell}))$ provided $m(\ell, \widetilde{\ell})$ is a standard multiplicity, i.e. if $\ell \in \left]\ell_{\min} = -\frac{m_s}{2}, \ell_{\max} = \frac{m_s}{2} + m_l\right[$ and $\widetilde{\ell} \geq -\frac{m_m}{2}$.

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Symmetries of $F_{\ell,\widetilde{\ell},\lambda}$ in the parameter ℓ

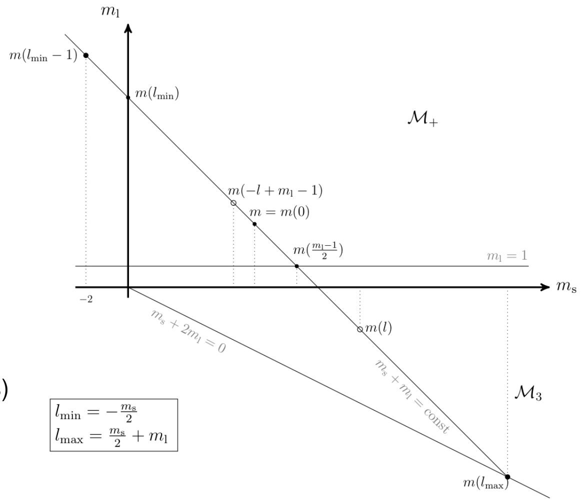
Lemma

For every m, ℓ and λ: $F_{\ell,\widetilde{\ell},\lambda}(m) = F_{-\ell+m_l-1,\widetilde{\ell},\lambda}(m)$

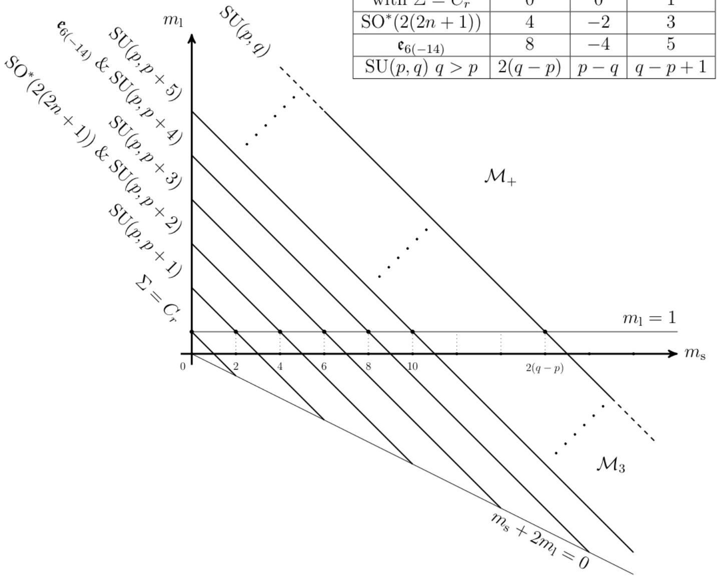
Geometric case:
$m_l = 1$, even in ℓ.

(we omit $\widetilde{\ell}$ from notation)

$\rightsquigarrow$ The estimates/asymptotics extend to $\ell \in]\ell_{\min} - 1, \ell_{\max}[$
(closed intervals for estimates)



G	m_s	ℓ_{min}	ℓ_{max}
with $\Sigma = C_r$	0	0	1
$SO^*(2(2n+1))$	4	-2	3
$\mathfrak{e}_{6(-14)}$	8	-4	5
$SU(p,q)$ $q > p$	$2(q-p)$	$p-q$	$q-p+1$



Bounded $(\ell, \widetilde{\ell})$-hypergeometric functions

Let $m = (m_{\mathrm{s}}, m_{\mathrm{m}}, m_{\mathrm{l}})$ be a nonnegative multiplicity.

$$\rho(m(\ell, \widetilde{\ell})) = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha(\ell, \widetilde{\ell})\alpha = \rho(m) - \frac{\ell}{2} \sum_{j=1}^{r} \beta_j + \frac{\widetilde{\ell}}{2} \sum_{1 \le i < j \le r} (\beta_j \pm \beta_i)$$

In particular,

$$\rho(m(0, 2\widetilde{\ell})) = \rho(m) + \widetilde{\ell} \sum_{1 \le i < j \le r} (\beta_j \pm \beta_i)$$

Since $\sum_{1 \le i < j \le r} (\beta_j \pm \beta_i) = \sum_{j=1}^{r} 2(j-1)\beta_j$ is a sum of positive roots, then $\rho(m(0, 2\widetilde{\ell})) > \rho(m)$ if $\widetilde{\ell} > 0$.

$C(\rho(m(0, 2\widetilde{\ell})))$= the convex hull of the set $\{w\rho(m(0, 2\widetilde{\ell})) : w \in W\}$.

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Assume that $m_{\mathrm{l}} \ge 1$ and $\widetilde{\ell} \ge 0$, $\ell \in]\ell_{\min} - 1, \ell_{\max}[$. Then, $F_{\ell, \widetilde{\ell}, \lambda}(m)$ is bounded if and only if $\lambda \in C(\rho(m(0, 2\widetilde{\ell}))) + i\mathfrak{a}^*$.

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Thank you!

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