The Compression Semigroup of the Dual Vinberg Cone

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Dedicated to the memory of Professor Takaaki Nomura.

Abstract We investigate the semigroup associated with the dual Vinberg cone and prove its triple and Ol'shanskiĭ polar decompositions. Moreover, we show that the semigroup does not have the contraction property with respect to the canonical Riemannian metric on the cone.

Keywords Dual Vinberg cone · Compression semigroup · Triple and Ol'shanskiĭ polar decompositions

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1 Introduction and Preliminaries

Semigroups of transformations leaving invariant a given set is a well-known tool in various fields of mathematics, for example, invariant convex cone theory and geometric control theory. In Lie group setting, probably the most important compression semigroups come from the Ol'shanskiĭ semigroups, i.e., compression semigroups of symmetric spaces $G_{\mathbb{C}}/G$, where G is a Hermitian Lie group. One extremely use-

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ful structure property of such semigroups is the existence and uniqueness of the Ol'shanskiĭ polar decomposition $G\exp(iC)$, where C is a convex cone in the Lie algebra of G which is invariant under the adjoint action of G. This decomposition has many applications to representations theory, see for example [4, 11, 12].

A compression semigroup associated naturally to an Euclidean Jordan algebra E was introduced in [6]. It is the compression semigroup of a symmetric cone $\mathcal{C}$ (the open cone of invertible squares in E), $\Gamma := \{g \in Co(E) \mid g(\mathcal{C}) \subset \mathcal{C}\}$, where $Co(E)$ is the conformal group of E. This semigroup Γ satisfies the Ol'shanskiĭ polar decomposition and, in addition, Γ admits a triple decomposition. Furthermore, elements of Γ are proved to be contractions for the invariant Riemannian metric on $\mathcal{C}$ [6, 7] and also for the Hilbert metric [8] and the Finsler metric [10]. The contraction property has many applications, for example, in Kalman Filtering theory (for the Hamiltonian semigroup) [1].

The purpose of this article is to study the compression semigroup of a homogeneous non-symmetric convex cone, which gives a new example of Lie semigroups admitting both the Ol'shanskiĭ polar decomposition and a triple decomposition, but does not have the contraction property with respect to the canonical metric. More precisely, the homogeneous cone Ω is given by

$$\Omega := \left\{x \in \mathbb{R}^5 \mid x_1 > 0,\ x_2 > 0,\ x_1x_2x_3 - x_1x_5^2 - x_2x_4^2 > 0\right\},$$

which is called the *dual Vinberg cone* [13, 14].

Let us first summarize some well-known facts about the real symplectic group and the symplectic semigroup, which will be utilized frequently in the investigation of the cone Ω. Let $\mathrm{Sym}(3,\mathbb{R})$ denote the space of 3×3 real symmetric matrices, and $\mathrm{Sym}^+(3,\mathbb{R})$ (resp. $\mathrm{Sym}^{++}(3,\mathbb{R})$) the subset of positive (resp. positive definite) matrices. Then, $\mathrm{Sym}^{++}(3,\mathbb{R})$ is a symmetric cone in the Euclidean Jordan algebra $\mathrm{Sym}(3,\mathbb{R})$ with the inner product given by $(x|y) = \mathrm{tr}\, xy$. Denote by Δ_1, Δ_2 and Δ_3 the principal minors of matrices in $\mathrm{Sym}(3,\mathbb{R})$. For a matrix M, denote by M^T its transpose, and if M is invertible, M^{-T} will denote $(M^T)^{-1}$.

Recall the symplectic group $\mathrm{Sp}(6,\mathbb{R}) = \{g \in \mathrm{GL}(6,\mathbb{R}) \mid gJg^T = J\}$ with $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. In a block form, an element $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}(6,\mathbb{R})$ with $A, B, C, D \in \mathrm{Mat}(3,\mathbb{R})$ belongs to $\mathrm{Sp}(6,\mathbb{R})$ if and only if

$$\begin{aligned} &A^TC\,,\ D^TB \in \mathrm{Sym}(3,\mathbb{R}),\\ &D^TA - B^TC = I, \end{aligned} \tag{1.1}$$

or equivalently

$$\begin{aligned} &BA^T, CD^T \in \mathrm{Sym}(3,\mathbb{R}),\\ &AD^T - BC^T = I. \end{aligned} \tag{1.2}$$

Lemma 1.1 *An element* $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(6,\mathbb{R})$ *has a unique triple decomposition*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & v \\ 0 & I \end{pmatrix} \begin{pmatrix} L & 0 \\ 0 & L^{-T} \end{pmatrix} \begin{pmatrix} I & 0 \\ u & I \end{pmatrix} \tag{1.3}$$

with $u, v \in \mathrm{Sym}(3, \mathbb{R})$ *and* $L \in GL(3, \mathbb{R})$ *if and only if D is invertible, and in this case*

$$L = D^{-T} = A - BD^{-1}C, \ v = BD^{-1} \ \textit{and} \ u = D^{-1}C. \tag{1.4}$$

It is well known that the symplectic group $\mathrm{Sp}(6, \mathbb{R})$ acts on the Siegel upper half space $T_{\mathrm{Sym}^{++}(3,\mathbb{R})} := \mathrm{Sym}(3, \mathbb{R}) + i\,\mathrm{Sym}^{++}(3, \mathbb{R})$ by linear fractional transformations, that is,

$$g \cdot z = (Az + B)(Cz + D)^{-1}, \quad \text{where } g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(6, \mathbb{R}), z \in T_{\mathrm{Sym}^{++}(3,\mathbb{R})}$$

which induces an isomorphism from $\mathrm{Sp}(6, \mathbb{R})/\{\pm I\}$ onto the holomorphic automorphism group $G(T_{\mathrm{Sym}^{++}(3,\mathbb{R})})$ of $T_{\mathrm{Sym}^{++}(3,\mathbb{R})}$. Since $\mathrm{Sym}(3, \mathbb{R})$ is the Šilov boundary of $T_{\mathrm{Sym}^{++}(3,\mathbb{R})}$, the action of $\mathrm{Sp}(6, \mathbb{R})$ is extended on $\mathrm{Sym}(3, \mathbb{R})$ (precisely, one should consider a conformal compacitification of $\mathrm{Sym}(3, \mathbb{R})$ on which the actions of all the elements $g \in \mathrm{Sp}(6, \mathbb{R})$ are well-defined [5]). In this action, we consider the compression semigroup (called also the symplectic semigroup) of the symmetric cone $\mathrm{Sym}^{++}(3, \mathbb{R})$,

$$\Gamma_{\mathrm{Sp}} := \left\{ g \in \mathrm{Sp}(6, \mathbb{R}) \mid g \cdot \mathrm{Sym}^{++}(3, \mathbb{R}) \subset \mathrm{Sym}^{++}(3, \mathbb{R}) \right\}$$

which is a closed subsemigroup of $\mathrm{Sp}(6, \mathbb{R})$.

It was proved in [6] that Γ_{Sp} can be given by

$$\Gamma_{\mathrm{Sp}} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(6, \mathbb{R}) \mid D \in \mathrm{GL}(3, \mathbb{R}), \ CD^T, D^T B \in \mathrm{Sym}^{+}(3, \mathbb{R}) \right\} \tag{1.5}$$

and has a triple decomposition $\Gamma_{\mathrm{Sp}} = \Gamma_{\mathrm{Sp}}^{+} G(3, \mathbb{R}) \Gamma_{\mathrm{Sp}}^{-}$, where

$$\Gamma_{\mathrm{Sp}}^{+} := \left\{ \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \mid B \in \mathrm{Sym}^{+}(3, \mathbb{R}) \right\},$$
$$G(3, \mathbb{R}) := \left\{ \begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix} \mid A \in \mathrm{GL}(3, \mathbb{R}) \right\},$$
$$\Gamma_{\mathrm{Sp}}^{-} := \left\{ \begin{pmatrix} I & 0 \\ C & I \end{pmatrix} \mid C \in \mathrm{Sym}^{+}(3, \mathbb{R}) \right\}.$$

It was also proved that the symplectic semigroup satisfies the following Ol'shanskiĭ polar decomposition $\Gamma_{\mathrm{Sp}} = G(3, \mathbb{R}) \exp(C_{\mathrm{Sp}})$, where C_{Sp} is the closed convex cone

$$C_{\mathrm{Sp}} := \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \in \mathfrak{sp}(6, \mathbb{R}) \mid B, C \in \mathrm{Sym}^{+}(3, \mathbb{C}) \right\},$$

and $\mathfrak{sp}(6, \mathbb{R})$ is the Lie algebra of $\mathrm{Sp}(6, \mathbb{R})$, that is,

$$\begin{aligned}\mathfrak{sp}(6, \mathbb{R}) &= \{X \in M(3, \mathbb{R}) \mid XJ + JX^T = 0\} \\ &= \left\{ \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \mid A \in \mathrm{Mat}(3, \mathbb{R}),\ \ B, C \in \mathrm{Sym}(3, \mathbb{R}) \right\}.\end{aligned}$$

Now we turn to the dual Vinberg cone Ω. Let V be the subspace of $\mathrm{Sym}(3, \mathbb{R})$ defined by

$$V := \left\{ x = \begin{pmatrix} x_1 & 0 & x_4 \\ 0 & x_2 & x_5 \\ x_4 & x_5 & x_3 \end{pmatrix},\ x_1, \dots, x_5 \in \mathbb{R} \right\}.$$

Then Ω is naturally identified with the intersection $\mathrm{Sym}^{++}(3, \mathbb{R}) \cap V$, that is,

$$\begin{aligned}\Omega &= \{x \in V \mid \Delta_1(x) > 0, \Delta_2(x) > 0, \Delta_3(x) > 0\} \\ &= \{x \in V \mid x \text{ is positive definite}\}.\end{aligned}$$

Let $T_\Omega := V + i\Omega \subset V_{\mathbb{C}}$ be the tube domain over Ω, $G(T_\Omega)$ the identity component of the holomorphic automorphism group on the tube domain T_Ω, and Γ the compression semigroup

$$\Gamma := \{g \in G(T_\Omega) \mid g \cdot \Omega \subset \Omega\} \tag{1.6}$$

of Ω. This semigroup Γ is a main object of the present work.

Here, we explain the organization of this paper. In Sect. 2, we describe the group $G(T_\Omega)$ as a subgroup of $\mathrm{Sp}(6, \mathbb{R})$. Then, we give a characterization of Γ as a subset of $G(T_\Omega)$ using the triple decomposition in Sect. 3. In Sect. 4, we show that Γ also admits an Ol'shanskiĭ polar decomposition. Finally, in Sect. 5, we show that Γ does not have a contraction property with respect to the canonical Riemannian metric on Ω.

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2 The Holomorphic Automorphism Group of T_Ω

First we shall determine the linear automorphism group $G(\Omega) := \{g \in GL(V) \mid g \cdot \Omega = \Omega\}$ of the cone Ω. Define

$$H := \left\{ A = \begin{pmatrix} a_1 & & \\ 0 & a_2 & \\ a_4 & a_5 & a_3 \end{pmatrix} \mid a_1, \ldots, a_5 \in \mathbb{R},\ a_1a_2 \neq 0,\ a_3 > 0 \right\},$$

and let H^+ be the subset of H consisting of diagonal matrices with positive entries. Then H forms a Lie group, and H^+ is its identity component. Let $\rho : H \to GL(V)$ be the representation of H given by $\rho(A)x := AxA^T \quad (A \in H,\ x \in V)$. Then H^+ as well as H acts transitively on the cone $\Omega \subset V$ by ρ. In other words, we have $\rho(H) \subset G(\Omega)$ and $\Omega = \rho(H^+)I_3$. For a parameter $(s_1, s_2, s_3) \in \mathbb{C}^3$, let $\Delta_{(s_1,s_2,s_3)}$ be the function on Ω given by

$$\begin{aligned}\Delta_{(s_1,s_2,s_3)}(x) &:= \Delta_1(x)^{s_1-s_2}\Delta_2(x)^{s_2-s_3}\Delta_3(x)^{s_3} \\ &= x_1^{s_1-s_3}x_2^{s_2-s_3}(\mathbf{det}\, x)^{s_3} \quad (x \in \Omega).\end{aligned}$$

The function $\Delta_{(s_1,s_2,s_3)}$ is relatively invariant under the action of H^+:

$$\Delta_{(s_1,s_2,s_3)}(\rho(A)x) = a_1^{2s_1}a_2^{2s_2}a_3^{2s_3}\Delta_{(s_1,s_2,s_3)}(x) \quad (x \in \Omega,\ A \in H^+). \tag{2.1}$$

Indeed, this equality characterizes the function $\Delta_{(s_1,s_2,s_3)}$ up to a constant multiple.

Let $\Omega^* \subset V$ be the dual cone of Ω. Namely, $\Omega^* := \{\xi \in V \mid (x|\xi) > 0 \text{ for all } x \in \overline{\Omega} \setminus \{0\}\}$. The so-called *Köcher–Vinberg characteristic function* φ_Ω of Ω is defined by $\varphi_\Omega(x) := \int_{\Omega^*} e^{-(x|\xi)}\, d\xi$ for $x \in \Omega$. It is known (see [3, Proposition I.3.1]) that, for any $g \in G(\Omega)$, we have

$$\varphi_\Omega(gx) = |\operatorname{Det}\, g|^{-1}\varphi_\Omega(x) \quad (x \in \Omega). \tag{2.2}$$

For $A = \operatorname{diag}(a_1, a_2, a_3) \in H$ and $x \in V$, we observe that

$$\rho(A)x = \begin{pmatrix} a_1^2x_1 & 0 & a_1a_3x_4 \\ 0 & a_2^2x_2 & a_2a_3x_5 \\ a_1a_3x_4 & a_2a_3x_5 & a_3^2x_3 \end{pmatrix},$$

so that $\operatorname{Det}\ \rho(A) = a_1^3a_2^3a_3^4$. For a general $A \in H^+$, because of the factorization $A = \operatorname{diag}(a_1, a_2, a_3)A'$ with a unipotent $A' \in H^+$, we have again $\operatorname{Det}\ \rho(A) = a_1^3a_2^3a_3^4$. Therefore, comparing (2.1) and (2.2), we conclude that there exists a constant $C > 0$ for which

$$\varphi_\Omega(x) = C\Delta_{(-3/2,-3/2,-2)}(x) = Cx_1^{1/2}x_2^{1/2}(\mathbf{det}\, x)^{-2}. \tag{2.3}$$

Let $G(\Omega)_{I_3}$ be the isotropy subgroup of $G(\Omega)$ at $I_3 \in \Omega$, and take $g \in G(\Omega)_{I_3}$. In general, for a function F on Ω, we denote by g^*F the pullback $F \circ g$. Since $G(\Omega)_{I_3}$ is a compact group, we have $|\operatorname{Det}\ g| = 1$, so that $g^*\varphi_\Omega^2 = \varphi_\Omega^2$ thanks to (2.2). Thus, by the uniqueness of irreducible factorization of the rational function φ_Ω^2, we have

$$g^*x_1 = C_1x_1, \quad g^*x_2 = C_2x_2, \quad g^*\mathbf{det}\, x = C_3\mathbf{det}\, x \tag{2.4}$$

or

$$g^*x_1 = C_1x_2, \quad g^*x_2 = C_2x_1, \quad g^*\mathbf{det}\, x = C_3\mathbf{det}\, x \tag{2.5}$$

with $C_1C_2C_3 = 1$. On the other hand, since $g \cdot I_3 = I_3$, we have $C_1 = C_2 = C_3 = 1$. Let us consider the case (2.4). We have $g^*\mathbf{det}\, x = \mathbf{det}\, x$, which means that

$$x_1x_2(g^*x_3) - x_1(g^*x_5)^2 - x_2(g^*x_4)^2 = x_1x_2x_3 - x_1x_5^2 - x_2x_4^2. \tag{2.6}$$

From this equality, we deduce $g^*x_5 = \pm x_5 + \alpha x_2$ with some $\alpha \in \mathbb{R}$. In fact, if g^*x_5 would contain other terms, for instance γx_3, then the left-hand side should contain the term of $x_1x_3x_5$, which does not appear in the right-hand side. By the same argument, we have $g^*x_4 = \pm x_4 + \beta x_1$ with some $\beta \in \mathbb{R}$. Actually, we have (2.6) in this case with

$$g^*x_3 = x_3 + \beta^2 x_1 + \alpha^2 x_2 \pm 2\beta x_4 \pm 2\alpha x_5.$$

On the other hand, since $g \cdot I_3 = I_3$, we have $\alpha = \beta = 0$. Therefore we conclude that $g = \rho(\mathrm{diag}(\pm 1, \pm 1, 1))$.

Let us turn to the case (2.5). Put $\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then $\rho(\sigma) : x = (x_1, \ldots, x_5) \mapsto (x_2, x_1, x_3, x_5, x_4)$ belongs to $G(\Omega)_{I_3}$ satisfying (2.5). Furthermore, if $g \in G(\Omega)_{I_3}$ satisfies (2.5), then $g \circ \rho(\sigma)$ satisfies (2.4). Now we conclude that

Lemma 2.1 *The isotropy subgroup $G(\Omega)_{I_3}$ is a finite group of order 8 generated by $\rho(\mathrm{diag}(-1, 1, 1))$, $\rho(\mathrm{diag}(1, -1, 1))$, and $\rho(\sigma)$.*

Corollary 2.1 *One has $G(\Omega) = \rho(H^+) \rtimes G(\Omega)_{I_3}$.*

We extend the action of $G(\Omega)$ to T_Ω by $g(z) = g(x) + ig(y)$, $z = x + iy$. The translation $t_v : z \mapsto z + v$ by $v \in V$ is a holomorphic automorphism of T_Ω, and the group N^+ of all such translations is an Abelian group isomorphic to V. The rational map s on T_Ω defined by

$$s : T_\Omega \ni z \mapsto \begin{pmatrix} -\frac{1}{z_1} & 0 & \frac{z_4}{z_1} \\ 0 & -\frac{1}{z_2} & \frac{z_5}{z_2} \\ \frac{z_4}{z_1} & \frac{z_5}{z_2} & \frac{\mathbf{det}\, z}{z_1z_2} \end{pmatrix} \in T_\Omega$$

belongs to $G(T_\Omega)$. Note that $s^2 = \rho(\mathrm{diag}(-1, -1, 1)) \neq \mathrm{Id}$, so that s is not an involution, but $s^4 = Id$. Let V' be the subspace of V defined by

$$V' := \left\{ u = \begin{pmatrix} u_1 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid u_1, u_2 \in \mathbb{R} \right\}.$$

For any $u \in V'$, let $\tilde{t}_u = s \circ t_u \circ s$ and denote by N^- the subgroup of $G(T_\Omega)$ of these transformations.

Keeping in mind the inclusion $T_\Omega \subset T_{\mathrm{Sym}^{++}(3,\mathbb{R})}$, we shall realize the group $G(T_\Omega)$ as a subgroup of $G(T_{\mathrm{Sym}^{++}(3,\mathbb{R})})$ (see Theorem 2.2). In other words, we shall see that any $g \in G(T_\Omega)$ can be described by an element of $\mathrm{Sp}(6,\mathbb{R})$. For $A \in H$, the corresponding $\rho(A) \in G(\Omega)$ is induced by the matrix $\begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix} \in G(3,\mathbb{R}) \subset \mathrm{Sp}(6,\mathbb{R})$. For $v \in V$, we identify the translation $t_v : T_\Omega \ni z \mapsto z + v \in T_\Omega$ with the matrix $\begin{pmatrix} I & v \\ 0 & I \end{pmatrix} \in \mathrm{Sp}(6,\mathbb{R})$. In this way, we regard $G_0 := \rho(H)$ and N^+ as subgroups of $\mathrm{Sp}(6,\mathbb{R})$. On the other hand, from a straightforward calculation, we see that the map s corresponds to the matrix

$$\begin{pmatrix} 0 & & & -1 & & \\ & 0 & & & -1 & \\ & & 1 & & & 0 \\ 1 & & & 0 & & \\ & 1 & & & 0 & \\ & & 0 & & & 1 \end{pmatrix} \in \mathrm{Sp}(6,\mathbb{R}).$$

Then, an easy matrix calculation tells us that the transform $\tilde{t}_u = s t_u s^{-1} \in N^-$ corresponds to $\begin{pmatrix} I & 0 \\ -u & I \end{pmatrix} \in \mathrm{Sp}(6,\mathbb{R})$.

Put $p_0 = iI \in T_\Omega$ and let $K = \{g \in G(T_\Omega) \mid g \cdot p_0 = p_0\}$ be the isotropy subgroup of $G(T_\Omega)$ at the point p_0.

Lemma 2.2 (cf.[2, Lemma 4.1]) *One has*

$$K = \left\{ k_{\theta,\phi} = \begin{pmatrix} C_{\theta,\phi} & -S_{\theta,\phi} \\ S_{\theta,\phi} & C_{\theta,\phi} \end{pmatrix} \mid \theta, \phi \in [0, 2\pi) \right\},$$

where

$$C_{\theta,\phi} = \begin{pmatrix} \cos\theta & & \\ & \cos\phi & \\ & & 1 \end{pmatrix}, \quad S_{\theta,\phi} = \begin{pmatrix} \sin\theta & & \\ & \sin\phi & \\ & & 0 \end{pmatrix}.$$

Theorem 2.1 *The group $G(T_\Omega)$ is generated by G_0, N^+ and s.*

Proof Let us take any $g \in G(T_\Omega)$ and put $z = g \cdot p_0$. Since $y = \Im z \in \Omega$, we can find $A \in H$ for which $\rho(A) \cdot I = y$. Putting $x = \Re z \in V$, we have $g \cdot p_0 = z = t_x \rho(A) \cdot p_0$, so that $k = \rho(A)^{-1} t_x^{-1} g$ belongs to K. Since $g = t_x \rho(A) k$, it is enough to show that k is generated by N^+, G_0 and s.

By Lemma 2.2, we have $k = k_{\theta,\phi}$ with some $\theta, \phi \in [0, 2\pi)$. First we consider the case $\theta = \phi$. If $\theta \neq \frac{\pi}{2}, \frac{3\pi}{2}$, then $\mathbf{det}\, C_{\theta,\theta} = \cos^2\theta \neq 0$, and we have

$$k_{\theta,\theta} = t_v \rho(A) \tilde{t}_{-u} \in N^+ G_0 N^-$$

with

$$A = (C_{\theta,\theta})^{-T} \in H, \quad u = (C_{\theta,\theta})^{-1} S_{\theta,\theta} \in V', \quad v = -S_{\theta,\theta}(C_{\theta,\theta})^{-1} \in V$$

thanks to Lemma 1.1. Thus $k_{\theta,\theta}$ is generated by N^+, G_0 and s in this case. For the case that $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$, the element $k_{\theta,\theta}$ equals s and s^{-1}, respectively, so that the claim holds for these cases, too.

Now we consider a general $k_{\theta,\phi}$. We can take an appropriate $\alpha \in \mathbb{R}$ for which $\mathbf{det}\, C_{\theta+\alpha,\,\phi+\alpha} \neq 0$. Similarly to the argument above, we see from Lemma 1.1 that $k_{\theta+\alpha,\phi+\alpha} \in N^+ G_0 N^-$. Finally, we have $k_{\theta,\phi} = k_{-\alpha,-\alpha} k_{\theta+\alpha,\phi+\alpha}$, which completes the proof. □

We remark that G_0 is not equal to the whole group $G(\Omega)$, but is a subgroup of $G(\Omega)$ of index 2. Indeed, $\rho(\sigma) \in G(\Omega) \setminus G_0$, and $\rho(\sigma)$ is a holomorphic automorphism on T_Ω but not an element of $G(T_\Omega)$. We also note that G_0 is not connected. Its identity component is $\rho(H^+)$.

Let us give another explicit description of the group $G(T_\Omega)$ as a subgroup of $\mathrm{Sp}(6, \mathbb{R})$. We set

$$W := \left\{ \begin{pmatrix} x_1 & 0 & x_6 \\ 0 & x_2 & x_7 \\ x_4 & x_5 & x_3 \end{pmatrix} \mid x_1, \ldots, x_7 \in \mathbb{R} \right\},$$

$$H' := \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ a_4 & a_5 & a_3 \end{pmatrix} \mid a_1, \ldots, a_5 \in \mathbb{R},\ a_3 > 0 \right\},$$

Then, we have

$$A,\ B \in H' \Rightarrow AB \in H', \tag{2.7}$$

$$A \in H',\ w \in W \Rightarrow Aw,\ wA^T \in W, \tag{2.8}$$

$$A \in H',\ u \in V' \Rightarrow uA,\ A^T u \in V', \tag{2.9}$$

and

$$A \in H',\ u \in V',\ w \in W \Rightarrow A + wu \in H'. \tag{2.10}$$

Define

$$G := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(6, \mathbb{R}) \mid A \in H',\ B \in W,\ C \in V',\ D^T \in H' \right\}. \tag{2.11}$$

Let us check that G is a subgroup of $\mathrm{Sp}(6, \mathbb{R})$. For two elements $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $g' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$ of G, we have

$$gg' = \begin{pmatrix} AA' + BC' & AB' + BD' \\ CA' + DC' & CB' + DD' \end{pmatrix}.$$

Then, we see from (2.7) to (2.10) that

$$AA' + BC' \in H', \quad AB' + BD' \in W, \quad CA' + DC' \in V', \quad (CB' + DD')^T \in H',$$

so that $gg' \in G$. On the other hand, since $G \subset \mathrm{Sp}(6, \mathbb{R})$, we have

$$g^{-1} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} g^T \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix},$$

for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$, whence we see that $g^{-1} \in G$.

Theorem 2.2 *The linear fractional action of* $\mathrm{Sp}(6, \mathbb{R})$ *on the Siegel upper half plane* $T_{\mathrm{Sym}^{++}(3,\mathbb{R})}$ *induces an isomorphism from G onto* $G(T_\Omega)$.

Proof For $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$ and $z \in T_\Omega$, we obtain $Az + B \in W_{\mathbb{C}}$ by (2.8) and $(Cz + D)^T \in H'_{\mathbb{C}}$ by (2.10), so that we have $g \cdot z \in W_{\mathbb{C}}$ by (2.8). On the other hand, since $g \in \mathrm{Sp}(6, \mathbb{R})$ and $z \in T_{\mathrm{Sym}^{++}(3,\mathbb{R})}$, we have $g \cdot z \in T_{\mathrm{Sym}^{++}(3,\mathbb{R})}$. Thus, $g \cdot z \in T_\Omega = T_{\mathrm{Sym}^{++}(3,\mathbb{R})} \cap W_{\mathbb{C}}$, and we have a group homomorphism from G into $G(T_\Omega)$. Thanks to Theorem 2.1, the map is surjective because G contains the matrices corresponding to $t_v \in N^+ \ (v \in V)$, $\rho(A) \in G_0 \ (A \in H)$ and s. Let us show the injectivity. Take $g \in G$ such that $g \cdot z = z$ for all $z \in T_\Omega$. Then, $g \cdot p_0 = p_0$ together with $g \in \mathrm{Sp}(6, \mathbb{R})$ implies $g = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$ with $A + iB \in U(3)$. Since $g \in G$, we have $A \in H'$, $A^T \in H'$ and $B \in V'$. Thus, we get $A = C_{\theta,\phi}$ and $B = S_{\theta,\phi}$ with some $\theta, \phi \in [0, 2\pi)$. Let us consider $z \in T_\Omega$ with $z_1 = z_2 = i$. Then

$$g \cdot \begin{pmatrix} i & 0 & z_4 \\ 0 & i & z_5 \\ z_4 & z_5 & z_3 \end{pmatrix} = \begin{pmatrix} i & 0 & e^{i\theta} z_4 \\ 0 & i & e^{i\phi} z_5 \\ e^{i\theta} z_4 & e^{i\phi} z_5 & z_3 + z_4^2 e^{i\theta} \sin\theta + z_5^2 e^{i\phi} \sin\phi \end{pmatrix}.$$

Thus, $g \cdot z = z$ implies $\theta = \phi = 0$, so that $g = I$. □

We see from Theorem 2.2 that each $g \in G(T_\Omega)$ is uniquely extended to a linear fractional transform on $T_{\mathrm{Sym}^{++}(3,\mathbb{R})}$. Let us present one more description of the group G:

Proposition 2.1 *One has*

$$G = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(6, \mathbb{R}) \mid A \in H',\ D^T \in H',\ D^T B \in V,\ CD^T \in V' \right\}.$$

Proof Let G' be the right-hand side. By (1.1), (2.8), and (2.9), we have $G \subset G'$. To show the converse inclusion, we take $g \in G'$ with $\mathbf{det}\, D \neq 0$. By (2.8) and (2.9) again, we get $B = D^{-T}(D^T B) \in W$ and $C = (CD^T)D^{-T} \in V'$. Thus $g \in G$. Since both G and G' are closed subset of $\mathrm{Mat}(6, \mathbb{R})$, we obtain $G' \subset G$ by a closure argument. □

Let $\Upsilon \subset G(T_\Omega)$ be the set consisting of $g \in G(T_\Omega)$ such that there exist $v \in V$, $A \in H$ and $u \in V'$ for which $g = t_v \rho(A) \tilde{t}_u$. Identifying $G(T_\Omega)$ with G by Theorem 2.2, we get an explicit description of the set Υ.

Proposition 2.2 *One has*

$$\Upsilon = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G \mid \mathbf{det}\, D \neq 0 \right\}.$$

Therefore, Υ is an open dense subset of $G(T_\Omega)$.

Proof Let Υ' be the right-hand side. The inclusion $\Upsilon \subset \Upsilon'$ follows from Lemma 1.1. To show the converse inclusion, we take $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Upsilon'$. Then, we have the equality (1.3) and (1.4). In particular, $v = BD^{-1} \in \mathrm{Sym}(3, \mathbb{R})$ belongs to W by (2.8), so that we get $v \in V = \mathrm{Sym}(3, \mathbb{R}) \cap W$. On the other hand, we have $u = D^{-1}C \in V'$ by (2.9) and $L = D^{-T} \in H$. Thus, $g = t_v \rho(L) \tilde{t}_{-u} \in \Upsilon$ and the assertion is verified. □

3 The Triple Decomposition of Γ

We shall investigate decomposition structures of the compression semigroup Γ defined by (1.6). More precisely, we will prove that any element g of the semigroup Γ admits a *triple decomposition* $g = t_v \rho(A) \tilde{t}_{-u}$, which is unique by Lemma 1.1.

Consider the following two closed subsemigroups of Γ

$$\Gamma^+ := \{t_v \mid v \in \overline{\Omega}\} = \left\{ \begin{pmatrix} I & v \\ 0 & I \end{pmatrix} \mid v \in \overline{\Omega} \right\},$$

$$\Gamma^- := \{\tilde{t}_{-u} \mid u \in \overline{\Omega}\} = \left\{ \begin{pmatrix} I & 0 \\ u & I \end{pmatrix} \mid u \in \overline{\Omega} \cap V' \right\},$$

and

$$\Gamma^{+0} := \{t_v \mid v \in \Omega\} = \left\{ \begin{pmatrix} I & v \\ 0 & I \end{pmatrix} \mid v \in \Omega \right\},$$

$$\Gamma^{-0} := \{\tilde{t}_{-u} \mid u \in \Omega\} = \left\{ \begin{pmatrix} I & 0 \\ u & I \end{pmatrix} \mid u \in \Omega \cap V' \right\}.$$

The latest are two subsemigroups of the interior Γ^0 of Γ. Now we state our first main theorem.

Theorem 3.1 *The semigroup* Γ *is contained in* Υ. *Moreover, one has* $\Gamma = \Gamma^+ G_0 \Gamma^-$.

Proof First we observe that for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$ with

$$A = \begin{pmatrix} a_1 & & \\ 0 & a_2 & \\ a_4 & a_5 & a_3 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & 0 & b_6 \\ 0 & b_2 & b_7 \\ b_4 & b_5 & b_3 \end{pmatrix}, \quad C = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} d_1 & 0 & d_4 \\ & d_2 & d_5 \\ & & d_3 \end{pmatrix},$$

the equality $AD^T - BC^T = I$ implies

$$a_k d_k - b_k c_k = 1 \qquad (k = 1,\ 2). \tag{3.1}$$

Moreover, if $z' = g \cdot z \in V_{\mathbb{C}}$ with $z \in V_{\mathbb{C}}$, then

$$z'_k = \frac{a_k z_k + b_k}{c_k z_k + d_k} \qquad (k = 1,\ 2). \tag{3.2}$$

Now we suppose $g \notin \Upsilon$, which means $\mathbf{det}\, D = 0$ by Proposition 2.2. Since $D^T \in H'$, we have $d_1 = 0$ or $d_2 = 0$. If $d_1 = 0$, we have $c_1 = -\frac{1}{b_1} \neq 0$ by (3.1). Let us consider the case

$$z = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in V.$$

By (3.2), we have $z'_1 = -a_1 b_1 - \frac{b_1^2}{x_1}$, so that we can take $x_1 > 0$ for which $z'_1 < 0$. Then $z \in \Omega$ and $z' = g \cdot z \notin \Omega$, which imply that $g \notin \Gamma$. Similarly, we can show $g \notin \Gamma$ if $d_2 = 0$. Therefore, we conclude that $\Gamma \subset \Upsilon$.

Now take $g \in \Gamma$ and let $g = t_v \rho(A) \tilde{t}_{-u}$ $(u \in V',\ A \in H,\ v \in V)$ be a triple decomposition. Let $\{x_n\}_{n=1}^{\infty} \subset \Omega$ be a sequence converging to 0. Then, $\Omega \ni g \cdot x_n = v + \rho(A)\tilde{t}_{-u} x_n \to v$ as $n \to \infty$, so that we get $v \in \overline{\Omega}$. Thanks to (3.2), we have $z'_k = v_k + \frac{a_k^2 z_k}{z_k + u_k}$ $(k = 1, 2)$ for $z' = g \cdot z \in V_{\mathbb{C}}$ with $z \in V_{\mathbb{C}}$. In particular, if $u_1 < 0$, then $g \cdot x$ is not defined for

$$x = \begin{pmatrix} -u_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \Omega,$$

which contradicts $g \in \Gamma$. Therefore $u_1 \geq 0$. We see that $u_2 \geq 0$ similarly, which completes the proof of the theorem. □

As a consequence, we have

$$\Gamma_{\mathrm{Sp}} \cap G = \Gamma \quad \text{and} \quad \Gamma^0_{\mathrm{Sp}} \cap G = \Gamma^0. \tag{3.3}$$

Let us describe Γ in a matrix block form.

Proposition 3.1 *One has*

$$\Gamma = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G \mid \mathbf{det}\,(D) \neq 0,\ \ D^T B \in \overline{\Omega},\ \ C D^T \in \overline{\Omega} \cap V' \right\}.$$

Proof Let $g = t_v \rho(L) \tilde{t}_{-u}$ be the triple decomposition of $g \in \Gamma$. Thanks to (1.4), we have $u = D^{-1}C = D^{-1}(CD^T)D^{-T}$ and $v = BD^{-1} = D^{-T}(D^T B)D^{-1}$. Therefore, the assertion follows from Theorem 3.1. □

4 The Ol'shanskiĭ Polar Decomposition of Γ

We see from (2.11) that the Lie algebra $\mathfrak{g}$ of G equals the subalgebra of $\mathfrak{sp}(6, \mathbb{R})$ given by

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & v \\ u & -A^T \end{pmatrix} \mid A \in \mathfrak{h},\ \ v \in V,\ \ u \in V' \right\},$$

where $\mathfrak{h}$ is the Lie algebra of $H \subset \mathrm{GL}(3, \mathbb{R})$, that is,

$$\mathfrak{h} = \left\{ \begin{pmatrix} a_1 & & \\ 0 & a_2 & \\ a_4 & a_5 & a_3 \end{pmatrix} \mid a_1, \ldots, a_5 \in \mathbb{R} \right\}.$$

Then, $\mathfrak{g}$ is graded by $\mathrm{ad}(Z_0)$ with $Z_0 := \begin{pmatrix} I/2 & 0 \\ 0 & -I/2 \end{pmatrix} \in \mathfrak{g}$. Namely, if $\mathfrak{g}_k := \{ X \in \mathfrak{g} \mid [Z_0, X] = kX \}$, then $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with

$$\mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} \mid u \in V' \right\},\quad \mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix} \mid A \in \mathfrak{h} \right\},\quad \mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \mid v \in V \right\}.$$

Let

$$C := \left\{ \begin{pmatrix} 0 & v \\ u & 0 \end{pmatrix} \mid v \in \overline{\Omega},\ u \in \overline{\Omega} \cap V' \right\}.$$

Then C is an $Ad(G(\Omega))$-invariant closed convex cone in $\mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ which is proper ($C \cap -C = \{0\}$) and generating ($C^0 \neq \emptyset$). Its interior C^0 is the set of matrices with $v \in \Omega$ and $u \in \Omega \cap V'$.

Theorem 4.1 *The compression semigroup* Γ *has the following Ol'shanskiĭ polar decomposition*

$$\Gamma = G_0 \exp(C)$$

with $\Gamma^0 = G_0 \exp(C^0)$ *as interior.*

Proof Let us denote $S := G_0 \exp(C)$ and prove that $\Gamma = S$.

First by [9, 12] it follows that S is a closed subsemigroup of G. Further, it is clear that $\Gamma^+ = \exp(\bar{\Omega})$, $\Gamma^- = \exp(\bar{\Omega} \cap V')$ and G_0 are closed subsemigroups of S. Thus, we have $\Gamma \subset S$ by Theorem 3.1.

On the other hand, since G_0 and $\exp(C)$ are subsemigroups of Γ_{Sp}, we see that $G_0 \exp(C) \subset \Gamma_{\mathrm{Sp}}$. In addition, $G_0 \subset G$ and $\exp(C) \subset G$, so that $S = G_0 \exp(C)$ is contained in both G and Γ_{Sp}. Therefore, $S \subset \Gamma$ thanks to (3.3). □

5 A Counter-Example to the Contraction Property of Γ

On a proper open convex cone $\mathcal{C} \subset \mathbb{R}^n$, the second derivative of the logarithm of the Köcher–Vinberg characteristic function $\varphi_{\mathcal{C}}$ of $\mathcal{C}$ gives a canonical Riemannian metric (see [3, Sect. I.4], [14, Chap. I, Sect. 3]):

$$(v|v')_x := D_v D_{v'} \log \varphi_{\mathcal{C}}(x) \qquad (v, v' \in \mathbb{R}^n,\ \ x \in \mathcal{C}),$$

where D_v denotes the directional derivative in v. Thanks to the relative invariance of $\varphi_{\mathcal{C}}$ under the action of the linear automorphism group $G(\mathcal{C})$ (see (2.2)), the canonical metric is $G(\mathcal{C})$-invariant. In particular, if $\mathcal{C}$ is a symmetric cone, the metric makes $\mathcal{C}$ a Riemannian symmetric space. For example, if $\mathcal{C} = \mathrm{Sym}^{++}(3, \mathbb{R})$, then the metric is given by the formula

$$(v|v')_x = 2\mathrm{tr}\,(x^{-1} v x^{-1} v') \qquad (v, v' \in \mathrm{Sym}(3, \mathbb{R}),\ x \in \mathrm{Sym}^{++}(3, \mathbb{R})). \tag{5.1}$$

It is proved in [6, Sect. 5] that, if $\mathcal{C}$ is symmetric, the compression semigroup $\Gamma_{\mathcal{C}}$ of $\mathcal{C}$ has the contraction property with respect to the canonical metric, that is,

$$(J(g, x)v | J(g, x)v)_{g(x)} \leq (v|v)_x \qquad (g \in \Gamma_{\mathcal{C}}, x \in \mathcal{C},\ \ v \in \mathbb{R}^n),$$

where $J(g, x)$ stands for the Jacobi matrix of g at x. We shall see that it is no longer the case when $\mathcal{C}$ is the dual Vinberg cone Ω.

Recalling (2.3), we see that the canonical Riemannian metric on Ω is given by

$$(v|v')_x = -\frac{1}{2}\Big(\frac{v_1 v_1'}{x_1^2} + \frac{v_2 v_2'}{x_2^2}\Big) + 2\,\mathrm{tr}\,(x^{-1} v x^{-1} v') \qquad (v, v' \in V,\ \ x \in \Omega). \tag{5.2}$$

Now we consider the case $v = v' = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and $x = I_3$. Then $(v|v)_x = -\frac{1}{2} + 2 \times 4 = 7.5$. In view of Theorem 3.1, we put $g_0 := t_{v_0} \in \Gamma$ with $v_0 := \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1.01 \end{pmatrix} \in \Omega$. Then $g_0(x) = I_n + v_0 = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2.01 \end{pmatrix} \in \Omega$ and $J(g_0, x)v = v$ since g_0 is a translation. We observe

$$(J(g_0, x)v|J(g_0, x)v)_{g_0(x)} = -\frac{1}{8} + 2\Big(\frac{6.01}{3.02}\Big)^2 = 7.795\cdots > 7.5 = (v|v)_x.$$

This phenomenon is caused by a behavior of 'the extra term' $-\frac{1}{2}(\frac{v_1 v_1'}{x_1^2} + \frac{v_2 v_2'}{x_2^2})$ in (5.2), compared with (5.1). Actually, the decrease of the second term $2\,\mathrm{tr}(x^{-1}vx^{-1}v')$ is little (from 8 to $2\Big(\frac{6.01}{3.02}\Big)^2 = 7.920\cdots$), while the extra term increases from $-1/2$ to $-1/8$.

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